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THE SPATIAL STRUCTURE OF THE STRESS FIELD IN THE NEIGHBOURHOOD OF THE CORNER POINT OF A THIN PLATE*

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The bending of a plate with a rigidly clamped edge is studied. The contour which forms the boundary of the middle cross-section has a corner point whose aperture angle is 2α. The effects of the boundary layers, namely of the plane layer (near the smooth part of the side surface S_h) and the three-dimensional layer (near the characteristic line on S_h) layer are compared. When $\alpha < \pi$, the plane boundary layer predominates, while when $\alpha = \pi$ the three-dimensional layer predominates. The latter consists of a linear combination of two special solutions $Y^{\pm}$ of the homogeneous problem in a wedge of unit thickness. Several terms of the asymptotic expansion of the displacement and stress fields in a plate are described and the residues are estimated. In case of a re-entrant corner, the terms in the asymptotic representation of the plate deformation energy are determined, taking into account the general effect of the three-dimensionality. When compared with the case of a smooth side surface, expressions appear containing the squares of the coefficients of the singular terms of the Kirchhoff solution and factors in expansions of the solutions $Y^{\pm}$ at infinity.

1. Formulation of the problem. Let us assume that the middle cross-section $\Omega \subset \mathbf{R}^2$ of the plate is bounded by a simple contour $\partial\Omega$, smooth (class C^∞) everywhere except at the corner point 0 with aperture angle $2\alpha \in (0, 2\pi]$. We shall consider the three-dimensional problem of the "pure" bending of a plate $Q_h = \Omega \times (-{}^1/_2h, {}^1/_2h)$ clamped along the side surface

$$\mu\nabla_x \cdot \nabla_x u(h, x) + (\mu + \lambda)\nabla_x\nabla_x \cdot u(h, x) = 0, \ x \in Q_h \tag{1.1}$$

$$\sigma^{(3)}(u; h, y, \pm{}^1/_2h) = \pm{}^1/_2 p(y) e^{(3)}, \ y = (x_1, x_2) \in \Omega \tag{1.2}$$

$$u(h, x) = 0, \ x \in S_h = \partial\Omega \times (-{}^1/_2h, {}^1/_2h) \tag{1.3}$$

Here λ, μ are the Lamé coefficients, $e^{(j)}$ is the unit vector in $\mathbf{R}^3$, $u = (u_1, u_2, u_3)$ is the displacement vector, p is the transverse load, $\sigma^{(3)} = \sigma e^{(3)}$, $\sigma(u)$ is the stress tensor with Cartesian coordinates $\sigma_{jk}(u)$. We will scale the characteristic dimension of the region to unity, in which case h will be a small positive dimensionless parameter representing the relative thickness of the plate.

Next we shall determine the first few terms of the asymptotic representation as $h \to 0$ of the solution $u(h, x)$ of problem (1.1)-(1.3).

2. Preliminary data. We know that the principal term of the asymptotic approximation to the solution of the problem of the bending of a plate is the function $w^0 \in W_2^2(\Omega)$, satisfying the problem

$$D_1\Delta_y^2 w^0(y) = p(y), \ y \in \Omega; \ w^0(y) = (\partial\omega^0/\partial n)(y) = 0, \quad y \in \partial\Omega \tag{2.1}$$

$$D_1 = {}^1/_3\mu(\mu + \lambda)(2\mu + \lambda)^{-1}$$

in which D_1 represents the reduced $(h = 1)$ cylindrical rigidity of the plate, Δ_y is the Laplace operator and n is the outer normal. More accurately, the deflection w^0 is used to re-establish

*Prikl. Matem. Mekhan., 55, 4, 653-661, 1991 the three-dimensional displacement vector

$$u^0(h, x) = h^{-3}\Xi(h, h^{-1}x_3, \Delta_y)\, w^0(y) + hW^0(y, h^{-1}x_3) \tag{2.2}$$

while the differential operator $\Xi = (\Xi_1, \Xi_2, \Xi_3)$ and vector W^0 have the form

$$\Xi_3(h, z, \nabla_y)\, w = w + {}^1/_2\nu(1-\nu)^{-1}(z^2 - {}^1/_{12})\Delta_y w,\ \Xi_k(h, z, \nabla_y) w = \tag{2.3}$$
$$hz\,\{h^2[{}^1/_6(1-\nu)^{-1}((2-\nu)z^2 + {}^1/_4(\nu-6))]\Delta_y - 1\}\,\partial w/\partial y_k$$
$$W_3{}^0(y, z) = [4\mu(1-\nu)]^{-1}\{{}^1/_2(3-\nu^2)(z^2 - 1/12) - (1-\nu^2)(z^4 - 1/80)\}p$$
$$W_k{}^0(y, z) = 0,\ k = 1, 2,\ \nu = \lambda[2(\mu+\lambda)]^{-1}$$

The stresses calculated in terms of the displacements (2.2) are:

$$\sigma_{jk}{}^0 = -12D_1h^{-2}x_3[\nu\delta_{j,k}\Delta_y + (1-\nu)\partial^2/\partial y_j\partial y_k]\, w^0 \tag{2.4}$$
$$\sigma_{j3}{}^0 = 6D_1h^{-1}(x_3{}^2h^{-2} - {}^1/_4)\,\partial\Delta_y w^0/\partial y_j\ (j, k = 1, 2)$$
$$\sigma_{33}{}^0 = {}^1/_2 x_3h^{-1}(3 - 4x_3{}^2h^{-2})\, p \tag{2.5}$$

($\delta_{j,k}$ is the Kronecker delta). The following estimate which follows from the asymptotically exact Korn inequality, substantiates the principal term of the asymptotic expression

$$\|(d+h)^{-2}(u_3 - u_3{}^0)\| + h^{-1}\|(d+h)^{-1}(u_j - u_j{}^0)\| + \|(d+h)^{-1}\nabla_y(u_3 - u_3{}^0)\| + \tag{2.6}$$
$$h^{-1}\|\nabla_y(u_j - u_j{}^0)\| + h\|(d+h)^{-1}\partial(u_3 - u_3{}^0)/\partial x_3\| +$$
$$\|(d+h)^{-1}\partial(u_j - u_j{}^0)/\partial x_3\| + h^{-1}\|\sigma_{jk}(u) - \sigma_{jk}^0\| +$$
$$\|(d+h)^{-1}(\sigma_{j3}(u) - \sigma_{j3}{}^0)\| + h^{-1}\|\sigma_{33}(u) - \sigma_{33}{}^0\| \leqslant ch^{-2}\|p; L_2(\Omega)\|$$

Here $\|\cdot\|$ is the norm in $L_2(Q_h)$, $d(y)$ is the distance between the point y and the boundary $\partial\Omega$. We stress that for the last two terms the estimate (2.6) provides very little information, and does not justify the asymptotic form of the stresses σ_{m3} $(m = 1, 2, 3)$ separated in (2.5). In other words, without additional assumptions concerning the differential properties of p and w, we cannot obtain the stresses σ_{m3} from the approximate formulas (2.5), and we must put $\sigma_{m3}^0 = 0$.

If the contour $\partial\Omega$ were smooth, then /1-5/ the solution of problem (1.1)-(1.3) could be expanded in the asymptotic series

$$u(h, x) \sim \sum_{j=0}^{\infty} h^{j-3}\{\Xi(h, h^{-1}x_3, \nabla_y)\, w^j(y) + \tag{2.7}$$
$$h^4W^j(y, h^{-1}x_3)\} + \chi(y)\sum_{q=2}^{\infty} h^{q-3}v^q(h^{-1}n, h^{-1}x_3, s)$$

in which w^j are solutions of problems of the form (2.1) with certain right-hand sides $z \mapsto W^j(y, z)$ is the vector function with zero average value over $z \in (-{}^1/_2, {}^1/_2)$ for all $y \in \Omega$, χ is a smooth truncated function equal to unity near $\partial\Omega$ and equal to zero outside the neighbourhood of the contour $\partial\Omega$ where the local coordinates n (normal) and s (tangential) are defined and v^q are solutions of the problem of the deformation of a half-strip with a clamped end (boundary layer-type terms) vanishing exponentially at infinity.

The second terms of the asymptotic form (for a smooth contour) were calculated in /6/. It was found that $v^0 = v^1 = 0$, and w^1 is a solution of the problem

$$\Delta_y{}^2 w^1(y) = 0,\ y \in \Omega;\ w^1(y) = 0,\ (\partial w^1/\partial n)(y) = c(\nu)\Delta_y w^0(y),\ y \in \partial\Omega, \tag{2.8}$$

where $c(\nu)$ is a quantity depending on Poisson's ratio ν only, and positive for $\nu \in (0, {}^1/_2)$ (the graph of the function $\nu \mapsto c(\nu)$ is shown in Fig.2 of /6/). The solution v^2 of the boundary layer-type is specified as follows:

$$v^2(\eta_1, \eta_2, s) = (X_n(\eta_1, \eta_2), X_3(\eta_1, \eta_2), 0)\,\Delta_y w^0(s, 0) \tag{2.9}$$

The vector X represents the solution of the problem of the plane deformation of the half-strip $\Pi = (0, +\infty) \times (-{}^1/_2, {}^1/_2)$ vanishing at infinity; there are no mass forces, the sides are stress-free and $X_n = -c(\nu)\eta_2$, $X_3 = {}^1/_2\nu(1-\nu)^{-1}(\eta_2{}^2 - {}^1/_{12}) - b(\nu)$ at the end $\{\eta : \eta_1 = 0,\ |\eta_2| < {}^1/_2\}$, $b(\nu)$ is a certain quantity (see Sect.3 of /6/ for more details). If we now take into account the named terms w^1, v^2 and denote the expressions analogous to (2.2) and (2.4), (2.5) by u^1 and $\sigma_{mn}{}^1$, respectively, then the inequality (2.6) will remain valid after replacing the superscript 0 by 1 and the right-hand side by $c_1h^{-1}(\|p\| + \|\nabla_y p\|)$ (in this case the estimate of the difference $\sigma_{33}(u) - \sigma_{33}^0)$ becomes meaningful).

In the case when a corner point lies on the boundary of the region, the relation $w^0 \in W_2^2(\Omega) \setminus W_2^3(\Omega)$ is possible by virtue of the singular nature of the solution at the corner tip. But in this case the expression σ_{j3}^0 from (2.5) will not belong to $L_2(Q_h)$ and this is absurd. As usual, in such situations an additional boundary layer appears near the rib $\omega = 0 \times (-{}^1/_2h,\ {}^1/_2h)$ at the side surface of the plate. This phenomenon is discussed in Sects.4 and 5, and the next section will deal with the formulation of the necessary results concerning the behaviour of solutions of plane problems (2.1) and (2.8) near the tip of the corner.

3. *A corner point.* We shall assume, for simplicity, that the load near the point O is zero. The power solutions of the Dirichlet problem for the biharmonic equation within the angle $K_\alpha = \{y : r > 0,\ |\theta| < \alpha\}$ have the form $r^{1+\Lambda_\pm}\Psi_\pm(\theta)$, where r, θ are polar coordinates and $\Lambda_\pm$ are the roots of the equation

$$\Lambda \sin 2\alpha \pm \sin 2\Lambda\alpha = 0 \tag{3.1}$$

(the roots $\Lambda_- = \pm 1$ for $\alpha \neq {}^1/_2\pi, \alpha_*, \pi$ and $\Lambda_\pm = 0$ are excluded from our discussion; here $\alpha_* \in ({}^1/_2\pi, {}^3/_4\pi)$ is a solution of the equation $\operatorname{tg} 2\alpha = 2\alpha$). The distribution of the roots of Eqs.(2.1) in the complex plane relative to the quantity α is known (see /17/ et al). If $\alpha \in (0, {}^1/_2\pi)$, then we have no roots within the strip $\Upsilon(1) = \{\lambda \in \mathbb{C}: |\operatorname{Re}\lambda| \leqslant 1\}$. If on the other hand $\alpha \in ({}^1/_2\pi, \pi]$, then $\Upsilon(1)$ contains a pair of real roots $\pm\Lambda_+(\alpha)$ and another pair $\pm\Lambda_-(\alpha)$. is added to them when $\alpha \in [\alpha_*, \pi]$. In what follows, we shall denote by $\Upsilon(l_\alpha)$ the largest strip containing only the roots of Eq.(3.1) named above.

The angular parts $\Psi_\pm$ are given by the equations

$$\begin{aligned}\Psi_+(\Lambda,\ \theta) &= C_+(\Lambda)\{\cos[(\Lambda+1)\alpha]\cos[(\Lambda-1)\theta] - \cos[(\Lambda-1)\alpha]\cos[(\Lambda+1)\theta]\}\\ \Psi_-(\Lambda,\ \theta) &= C_-(\Lambda)\{\sin[(\Lambda+1)\alpha]\sin[(\Lambda-1)\theta] - \sin[(\Lambda-1)\alpha]\sin[(\Lambda+1)\theta]\}\end{aligned} \tag{3.2}$$

The explicit form of the normalizing factors $C_\pm(\Lambda)$ will not be used (except in the case of $\alpha = \pi$ which will be discussed at the end of this section). We shall use only the following fact /8/: by virtue of the choice of the factors mentioned above the functions $U_\pm(y) = r^{1+\Lambda_\pm(\alpha)}\Psi_\pm(\Lambda_\pm(\alpha),\ \theta)$ and $Z_\pm(y) = r^{1-\Lambda_\pm(\alpha)}\Psi_\pm(-\Lambda_\pm(\alpha),\ \theta)$ can be normalized in the following manner:

$$\langle U_\pm, Z_\pm\rangle_\delta \equiv \delta \int_{-\alpha}^{\alpha} \Big\{U_\pm \frac{\partial}{\partial r}\Delta_y \bar{Z}_\pm - \frac{\partial U_\pm}{\partial r}\Delta_y \bar{Z}_\pm + \frac{\partial \bar{Z}_\pm}{\partial r}\Delta_y U_\pm - \bar{Z}_\pm \frac{\partial}{\partial r}\Delta_y U_\pm\Big\}\Big|_{r=\delta} d\theta = D_1^{-1} \tag{3.3}$$

(a bar denotes complex conjugation, and its presence is not essential for real $\Lambda_\pm(\alpha)$ or, in particular, when $\alpha \in ({}^1/_2\pi, \pi]$). We stress that the left-hand side of identity (3.3) is independent of $\delta > 0$.

If the angle α is small, then the solution w^0 will belong to $W_2^4(\Omega)$. When $\alpha \in (0, {}^1/_2\pi)$, inclusion $w^0 \in W_2^3(\Omega)$ will hold and the following representation will hold for $\alpha \in ({}^1/_2\pi, \pi]$:

$$w^0(y) = \sum_\pm c_\pm U_\pm(y) + O(r^{1+l_\alpha}), \quad r \to 0 \tag{3.4}$$

where we have assumed that $c_- = 0$ for $\alpha \in ({}^1/_2\pi, \alpha_*)$. In the case $\alpha \in ({}^1/_2\pi, \pi]$ we have a non-trivial solution $\zeta_+ \in W_2^1(\Omega)$ of the homogeneous problem (2.1), with the asymptotic representation

$$\zeta_\pm(y) = Z_\pm(y) + O(r^{1+\Lambda_+(\alpha)}), \quad r \to 0 \tag{3.5}$$

The same solution ζ_- will fall into $W_2^1(\Omega)$ if $\alpha \in (\alpha_*, \pi]$. The following formulas /8/ hold by virtue of the normalization of (3.3):

$$c_\pm = \int_\Omega p(y)\overline{\zeta_\pm(y)}\,dy \tag{3.6}$$

For a convex angle K_α the solution w^1 of problem (3.8) is contained within the space $W_2^2(\Omega)$. If on the other hand the angle is re-entrant (i.e. $\alpha > {}^1/_2\pi$), problem (2.8) will certainly have no solution belonging to $W_2^2(\Omega)$ when $c_+ \neq 0$. In a wider class $W_2^1(\Omega)$

we have no theorem of uniqueness and the solution has to be obtained from the asymptotic representation

$$w^1(y) = \sum_{\pm} c_{\pm} W_{\pm}(y) + O(r^{1+\Lambda_+(\alpha)}), \quad r \to 0 \tag{3.7}$$

Here $W_{\pm}(y) = r^{\Lambda_{\pm}(\alpha)} \Phi_{\pm}(\theta)$ is the solution of the problem within the angle K_α

$$\Delta_y^2 W_{\pm}(y) = 0, \; y \in K_\alpha$$
$$W_{\pm}(y) = 0, \; (\partial W_{\pm}/\partial n)(y) = c(\nu)\, \Delta_y U_{\pm}(y), \; y \in \partial K_\alpha \setminus 0 \tag{3.8}$$

According to the general results /9/ the solution (3.7) of problem (2.8) exists and is unique, and the model problem (3.8) has a solution of the form shown only if the number $\Lambda_{\pm}(\alpha) - 1$ does not coincide with one of the roots of Eq.(3.1). The latter demand is satisfied for $\alpha \in (1/2\pi, \pi)$, and is violated when $\alpha = \pi$, since $\Lambda_{\pm}(\pi) = 1/2$. Therefore, the corner point with aperture angle 2π (a crack in the plate with clamped edges) needs additional discussion.

When $\alpha = \pi$, the special solutions $U_{\pm}$ and $Z_{\pm}$ will have the form

$$U_+(y) = Ar^{3/2}\psi_+(\theta), \quad U_-(y) = -Ar^{3/2}\psi_-(\theta), \quad A = (2\mu)^{-1}\sqrt{2} \tag{3.9}$$
$$Z_+(y) = 3Br^{1/2}\psi_+(\theta), \; Z_-(y) = -Br^{1/2}\psi_-(\theta), \; B = 3/4\pi^{-1}(1-\nu)\sqrt{2}$$
$$\psi_+(\theta) = \cos 1/2\theta + 1/3 \cos 3/2\theta, \; \psi_-(\theta) = \sin 1/2\theta + \sin 3/2\theta$$

(see e.g. /10, 7/, and compare with (3.2) and (3.3)). We can confirm directly that the following functions are solutions of the model problems (3.8): $W_+ = 0$ and

$$W_-(y) = \pi^{-1} c(\nu) A r^{1/2} [\psi_-(\theta) \ln r + \theta(\cos 1/2\,\theta - \cos 3/2\,\theta)] + A_0 r^{1/2} \psi_-(\theta) \tag{3.10}$$
$$A_0 = -(4\pi\mu)^{-1} c(\nu) \sqrt{2}$$

We note that solution (3.10) was obtained apart from the linear combination $A_0 Z_- + B_0 Z_+$. the coefficient B_0 was made equal to zero in order to preserve the property that the function is odd in the variable θ, and the coefficient A_0 was chosen from the condition $\langle U_-, W_- \rangle_\delta = -8c(\nu) A^2 \ln \delta$ (this relation was used in deriving formula (6.4)). Thus by virtue of /9/, after changing the form of the function W_- (a linear dependence on $\ln r$), all that was said about the solution w^1 in the case of $\alpha \in (-1/2\pi, \pi)$, also remains true for $\alpha = \pi$.

4. Boundary layer near the rib ω. We introduce near the point O the "rapid" variables $\eta = (\eta_1, \eta_2, \eta_3) = h^{-1}x$ and write formally $h = 0$. As a result, the plate Q_h will be transformed into a sector of the layer $K_\alpha \times (-1/2, 1/2)$. According to the method of matched asymptotic expansions we should construct the solutions $Y^{\pm}$ of the homogeneous problem of the theory of elasticity in the "wedge" $K_\alpha \times (-1/2, 1/2)$, with the following asymptotic representation at infinity:

$$Y^{\pm}(\eta) = e^{(3)} U_{\pm}(\eta') + o(\rho^{1+\Lambda_{\pm}(\alpha)}), \; \rho \to \infty$$
$$(\eta' = (\eta_1, \eta_2), \; \rho = |\eta'| = (\eta_1^2 + \eta_2^2)^{1/2}) \tag{4.1}$$

The solutions named above transform the elastic energy functional into infinity. From /11/ it follows that these solutions exist and are unique, and formula (4.1) admits of the following refinement:

$$Y^{\pm}(\eta) = \Xi(1, \eta_3, \partial/\partial\eta')\{U_{\pm}(\eta') + W_{\pm}(\eta') + \mu^{-1} k_{\pm}(\alpha, \nu) Z_{\pm}(\eta')\} +$$
$$+ \chi_0(\theta - \alpha)\, \mathrm{X}(\eta^{(\alpha)})\, \Delta_\eta U_{\pm}(\eta')|_{\theta=\alpha} +$$
$$\chi_0(\theta + \alpha)\, \mathrm{X}(\eta^{(-\alpha)})\, \Delta_\eta U_{\pm}(\eta')|_{\theta=-\alpha} + O(\rho^{\gamma_{\pm}(\alpha)}) \tag{4.2}$$

Before explaining the notation adopted in (4.2), we stress that the algorithm for constructing the expansions (4.2) is in fact the same as in the case of a thin plate. The fact is, that the form of the expansion of the solution of the elliptic problem in the region Ω is governed by the form of the set M_R cut out by the sphere $\{\eta: |\eta| = R\}$ from Ω. When $R \to \infty$, the set M_R should be interpreted as a thin region whose length is of the order of $O(R)$ and whose width is of the order of $O(1)$.

The first group of terms in (4.2) is completely analogous to the terms of the first series of (2.7). The operator Ξ is given by formula (2.3), $U_\pm, W_\pm$ and $Z_\pm$ are functions described in Sect.3 and $k_\pm(\alpha, \nu)$ is a quantity depending only on the aperture angle α of K_α and Poisson's ratio ν (in all probability $k_\pm(\alpha, \nu)$ can only be calculated using a digital computer).

The second group of terms in (4.2) represents the boundary layer-type solution: $\eta^{(\pm\alpha)} = (\pm\eta_1 \sin\alpha - \eta_2 \cos\alpha,\ \eta_3)$ are Cartesian coordinates in the planes parallel to the $O\eta_3$ axis and perpendicular to the side edges of the wedge, X is the solution of the problem in the half-strip Π appearing in (2.9) and vanishing exponentially at infinity, χ_0 is a smooth, even truncated function and $\chi_0(t) = 1$ near the point $t = 0$.

We shall deal separately with the problem of estimating the residue which includes terms of order $O(r^{1+\Lambda_\pm(\alpha)-2})$ and $O(r^{1-\Lambda_\pm(\alpha)-1})$ (smooth type solutions following the functions $U_\pm$, $W_\pm$ and $Z_\pm$, respectively), smaller terms of the boundary layer, and solutions of the form $\Xi r^{1-\Lambda}{}_\pm\Psi_\pm$ corresponding to the other roots $\Lambda_\pm$ of Eq.(3.1) (here $\operatorname{Re}\Lambda_\pm > \operatorname{Re}\Lambda_\pm(\alpha) > 0$). Thus the indices $\gamma_\pm(\alpha)$ are subject to the conditions

$$\gamma_\pm(\alpha) \geqslant \max\{\Lambda_\pm(\alpha) - 1,\ 1 - l_\alpha\} \tag{4.3}$$

If $\alpha \in (-{}^1/_2\pi, \pi]$ (or $\alpha \in [\alpha_*, \pi]$), then the first expression on the right-hand side of (4.2) will determine the principal term of the asymptotic form of the displacement field Y^+ or Y^-), and the solutions of boundary layer-type are $O(r^{\Lambda_\pm(\alpha)-1})$ and can be neglected. However, after a single differentiation, the last solutions will retain their order and will therefore be included in the principal part of the asymptotic expression (as $\rho \to \infty$) of the stress and deformation fields. In the case when $\alpha \in (0, {}^1/_2\pi)$ (or $\alpha \in (0, \alpha_*) \setminus \{{}^1/_2\pi\}$), the term containing Z_+ (or Z_-) can be eliminated from the expansion (4.2). Since $\operatorname{Re}\Lambda_\pm(\alpha) > 1$, it follows that since the value of the residue increases, the presence of the function $Z_\pm$ vanishing at infinity does not provide any information about the asymptotic form of the solution. It is clear that this term will be restored in the expansion after the minor terms have been taken into account. (The characteristics assigned to the quantities in (4.2) are identical with the further results of Sects.5 and 6, referring to the effect of the corner point on the stress-deformation state of the plate). Finally, we note that when the quantity $\Lambda_\pm(\alpha) - 2$ or $\Lambda_\pm(\alpha) - 1$ is identical with the root of Eq.(3.1), then an additional logarithmic multiplier may appear in the expression $\rho^{\gamma_\pm(\alpha)}$ (compare with (3.10)), which shall not show, which will increase the value of the exponent $\gamma_\pm(\alpha)$ by a small, positive amount (if there is no $\ln\rho$, then formula (4.3) will have the equality sign).

5. ***Matching solutions of different types.*** The effect of the three-dimensional nature of the stress-deformation state near the rib $\omega \subset S_h$ will be reduced to the fact that the solution $u(h, x)$ of problem (1.1)-(1.3) will be generally represented, in a small neighbourhood of ω, by the sum

$$Y(h, x) = c_+ h^{\Lambda_+(\alpha)-2} Y^+(h^{-1}x) + c_- h^{\Lambda_-(\alpha)-2} Y^-(h^{-1}x) \tag{5.1}$$

The coefficients $c_\pm$ are taken from the expansion (3.4) of the solution of the plane Kirchhoff problem (2.1). The presence in (5.1) of small multiplying factors $h^{1+\Lambda_\pm(\alpha)}$ shows that the degree of influence of the three-dimensional boundary layer can vary, depending on the magnitude of the angle α. For example, when $\alpha \in (0, {}^1/_2\pi)$ the two-term asymptotic expressions ($j = 0, 1$ and $q = 2$ in (2.7)) constructed in /6/ can be corrected by terms of the same series (2.7) for $j = 2$, $q = 3$, and the three-dimensional nature emerges in even lower terms.

In order to confirm what was said before and to clarify the degree of influence exerted by the boundary layer (5.1) in the case of $\alpha \in ({}^1/_2\pi, \pi]$, we must determine the global asymptotic approximation to the solution of initial problem. With this in mind, we shall employ a version of the method of matched asymptotic expansions (see /12, 13/ etc.).

We denote by $V^\circ(h, x)$ the sum of terms on the right-hand side of (2.7) with the indices $j = 0,1$ and $q = 2$. Let us first have $\alpha \in ({}^1/_2\pi, \pi)$ (we recall that formally it is assumed that $c_- = 0$ in (3.4) for $\alpha \in ({}^1/_2\pi, \alpha_*)$). Comparing relations (3.4), (3.7) and (2.9) with expansions (4.2) (the third term in curly brackets was transferred to the residue) we find, that within the intermediate zone $r \sim h^{1/2}$, the quantities $V^\circ(h, x)$ and $Y(h, x)$ will, on the whole, be identical. The difference will be detected only in the lower terms of the expansions. By taking into account in (4.2) terms containing $Z_\pm$, we are obliged to bring into the representation smooth-type solutions which have the following expressions serving as asymptotic representations as $r \to 0$:

$$\mu^{-1} c_\pm h^{\Lambda_\pm(\alpha)-2} k_\pm(\alpha, \nu) Z_\pm(h^{-1}y) = h^{2\Lambda_\pm(\alpha)-3} \mu^{-1} c_\pm k_\pm(\alpha, \nu) Z_\pm(y) \tag{5.2}$$

We recall the comments accompanying Eq.(3.5) indicating the solutions $\zeta_\pm \in W_2^1(\Omega)$ of the homogeneous problem (2.1), and put

$$V(h, x) = V^\circ(h, x) + \mu^{-1} \sum_{\pm} h^{2\Lambda_\pm(\alpha)-3} c_\pm k_\pm(\alpha, \nu)\, \Xi(h, h^{-1}x_3, \nabla_y)\, \zeta_\pm(y) \tag{5.3}$$

Now, by virtue of (3.5), V and Y will be matched using the last terms from within the braces in (4.2).

Let us now choose the global asymptotic approximation U. Since the solutions at the corner tip are singular, it follows that we must change the field (5.3) containing the terms Ξw^0, Ξw^1, $\Xi\zeta_\pm$ and v^2. In the function v^2 of the plane boundary layer-type given by the formula (2.9), we replace $\Delta_y w^0$ by the difference

$$\Delta_y w^0(y) - \chi_0(r) \sum_{\pm} c_\pm \Delta_y U_\pm(y) \tag{5.4}$$

The subtrahend removes the singularities of the solution w^0 isolated in (3.4), and their effect is transferred to the function (5.1) of three-dimensional boundary layer-type. Note also that, in order to make the expansions (5.3) and (5.1) compatible, we must make the choice of the cut χ in (2.7) more precise; near the point $y = 0$ the equality $\chi(y) = \chi_0(\theta - \alpha) + \chi_0(\theta + \alpha)$ must be observed (compare with (4.2)).

The expression Ξw^0 is determined in accordance with (2.3) and contains terms $N_k w^0$, where $N_k(\nabla_y)$ are differential operators of orders $k = 0, \ldots, 3$. If $k \leqslant 2$, we replace $N_k w^0$ and Ξw^0 by $N_k w^0 - \chi_0 \sum c_\pm N_k U_\pm$ analogous to (5.4). If on the other hand $k = 3$, we replace $N_k w^0$ by

$$(1 - \chi_0(h^{-1}r)) \left\{ N_k(\nabla_y) w^0(y) - \chi_0(r) \sum_{\pm} c_\pm N_k(\nabla_y) U_\pm(y) \right\} \tag{5.5}$$

In other words, just as before for v^2, some of the singular components of the field Ξw^0 will refer to the three-dimensional boundary layer. Moreover, the difference $\mathbf{w}^0 = w^0 - \chi_0 \sum c_\pm U_\pm$ belongs to $W_2^3(\Omega)$, but may not necessarily fall within $W_2^4(\Omega)$. In the latter case we cannot have the inclusion $N_3 \mathbf{w}^0 \in W_2^1(Q_h)$ which is necessary, but this can be corrected by multiplying by the cut-off which is zero for small $h^{-1}r$. We will treat the solutions w^1 and $\zeta_\pm$ in the same manner. By virtue of (3.7) and (3.5) the functions $\mathbf{w}^1 = w^1 - \chi_0 \sum c_\pm W_\pm$ and $\boldsymbol{\zeta}_\pm = \zeta_\pm - \chi_0 Z_\pm$ are contained within $W_2^2(\Omega)$, therefore we can prescribe for the expressions $N_k w^1$, $N_k \zeta_\pm$ a replacement of the type (5.4) for $k = 0, 1$ and of the type (5.5) for $k = 2, 3$.

Let us denote by $\mathbf{V}(h, x)$ the right-hand side of Eq.(5.3) in which the above transformation has been carried out, and by $\mathbf{Y}(h, x)$ the sum (5.1) multiplied by the truncation $\chi_0(r)$, and let us write $\mathbf{U} = \mathbf{V} + \mathbf{Y}$. This defines the global approximation to the solution of problem (1.1)-(1.3). We stress that the complex construction given here is needed only for the strict formulation of the estimate of the residue. Use of different expansions for different zones is asymptotically justified. Outside the small neighbourhood ω of the rib the approximation (5.3) is appropriate, and within this neighbourhood we use (5.1).

If we now substitute, as in /6, 14, 15/, the function U obtained earlier into the Eqs. (1.1)-(1.3), calculate and estimate the resulting discrepancy, and use the Korn inequality, we obtain the following relation:

$$\begin{gathered} \|(d+h)^{-2}(u_3 - \mathbf{U}_3)\| + h^{-1}\|(d+h)^{-1}(u_j - \mathbf{U}_j)\| + \|(d+h)\nabla_y(u_3 - \mathbf{U}_3)\| + \\ h^{-1}\|\nabla_y(u_j - \mathbf{U}_j)\| + h\|(d+h)^{-2}\partial(u_3 - \mathbf{U}_3)/\partial x_3\| + \|(d+h)^{-1}\partial(u_j - \mathbf{U}_j)/\partial x_3\| + \\ h^{-1}\|\sigma_{jk}(u - \mathbf{U})\| + \|(d+h)^{-1}\sigma_{j3}(u - \mathbf{U})\| + \|\sigma_{33}(u - \mathbf{U})\| \leqslant \\ ch^{-1}\{\|r^{2-l\alpha-\varepsilon}p\| + \|r^{1-l\alpha-\varepsilon}\nabla_y p\|\}, \quad j, k = 1, 2 \end{gathered} \tag{5.6}$$

in which $0 < \varepsilon$ is arbitrary. The weighting factors are included in the norm of the function p in order to ensure /9/ the validity of expansions (3.4) and (3.7). Estimates of the residues in the expansions should also be rewritten in terms of the Sobolev weight classes /9/. We will ignore this fact in order to simplify the presentation. Besides, the (non-obligatory) assumption concerning the load p given at the beginning of Sect.3, exhausts all questions.

In the case when $\alpha = \pi$ (a crack) the asymptotic form is modified slightly by virtue of the presence of $\ln\rho = \ln r - \ln h$ in the function $W_-(\eta')$ (see (3.10)). The form of the initial formula (5.3) will be changed thus:

$$\begin{gathered} V(h, x) = V^\circ(h, x) + \mu^{-1}h^{-2}\{c_+ k_+(\pi, \nu)\, \Xi(h, h^{-1}x_3, \nabla_y)\, \zeta_+(y) + \\ c_-[k_-(\pi, \nu) + {}^2/_3 c(\nu)(1-\nu)^{-1}\ln h]\, \Xi(h, h^{-1}x_3, \nabla_y)\, \zeta_-(y)\} \end{gathered} \tag{5.7}$$

All subsequent transformations will remain the same, but the factor h^{-1} from the right of (5.6) will now be written as $h^{-1-\varepsilon}$ (due to the appearance of logarithms; see Sect.4).

We will finally discuss the case $\alpha \in (0, {}^1/_2\pi)$, when the construction of the asymptotic form is simplified due to the possibility of transforming the boundary layer terms to the residue. If the solutions w^0 and w^1 are found, respectively, in the spaces $W_2^4(\Omega)$ and $W_2^3(\Omega)$ (the angle α is small), then the stresses calculated in terms of the functions w^0 and w^1 using the formulas (2.4), (2.5) (see Sect.5 of /6/) will belong to $L_2(Q_h)$. This means that the quantity (5.3) itself can be used as the asymptotic approximation and the inequality (5.6) will hold when $U = V$. If, on the other hand, w^0 lies in $W_2^3(\Omega)$ (when $\alpha \in (0, {}^1/_2\pi)$, this is always true) but not in $W_2^4(\Omega)$, then taking into account in (5.3) the contribution of the solution w^1, we must multiply by truncation $1 - \chi_0(h^{-1}r)$ the terms of expansion of the function $N_3 w^1$ emerging from the class $W_2^1(Q_h)$ (compare with (5.5)). The resulting error here is of the order of $o(h^{-1})$.

6. Discussion. If the side surface S_h is smooth, the effect of the plane boundary layer away from the edge of the plate will be reduced mainly to replacing the Kirchhoff solution $h^{-3}w^0$ by the sum $h^{-3}w^0 + h^{-2}w^1$, and the error of the next power will have the form $h^{-1}w^2$. From Eq.(5.3) it follows that an additional contribution of the corner point of the contour $\partial\Omega$ is made by the following approximation to the bending of the plate:

$$h^{-3}w^\circ + h^{-2}w^1 + \mu^{-1}\sum_{\pm} h^{2\Lambda_\pm(\alpha)-3} c_\pm k_\pm(\alpha, \nu)\,\zeta_\pm \tag{6.1}$$

When $\alpha \in (0, {}^1/_2\pi)$, the inequality $\operatorname{Re}\Lambda_\pm(\alpha) > 1$ holds, i.e. the third expression in (6.1) is weaker than the quantity $h^{-1}w^2$. In the case of $\alpha \in ({}^1/_2\pi, \pi)$ (a re-entrant angle) at least one of the indices $2\Lambda_\pm(\alpha) - 3$ is smaller than -1, and this means that the perturbation caused by the corner point is stronger than $h^{-1}w^2$. However, the term $h^{-2}w^1$, constructed in /6/, serves, as before, as the main correction. Finally, in case of a crack we add to the sum (6.1), in accordance with (5.7), the term ${}^2/_3 c_- c(\nu)[\mu(1-\nu)]^{-1}\zeta_- h^{-2}\ln h$, which in fact represents the main perturbation of the Kirchhoff solution $h^{-3}w^\circ$.

Let us find the first term of the asymptotic expression for the potential energy of the deformation of the plate

$$U_h(u) = E_h(u) - A_h = -{}^1/_2 A_h = -{}^1/_4 \sum_{\pm}\int_{\Omega} p(y)\, u_3(h, y \pm {}^1/_2 h)\, dy \tag{6.2}$$

Here $E_h(u)$ is the elastic energy functional and A_h is the work done by external forces. Substituting into (6.2) the approximation obtained for the solution of problem (1.1)-(1.3) and recalling the estimate (5.6) we conclude that the following relation holds for $\alpha \in ({}^1/_2\pi, \pi)$:

$$U_h(u) = -{}^1/_2 h^{-3}\int_{\Omega} p\left(w^\circ + hw^1 + \sum_{\pm}\mu^{-1}h^{2\Lambda_\pm(\alpha)} c_\pm k_\pm(\alpha, \nu)\,\zeta_\pm\right) dy + O(h^{-1})$$

Using Eqs.(2.1) and (2.8) and integrating them by parts in Ω and applying the relations (3.6), we obtain the following asymptotic formula:

$$U_h(u) = -{}^1/_2 h^{-3}\mathbf{E}_1(w^\circ) + {}^1/_2 h^{-2} c(\nu) D_1 \int_{\partial\Omega} |\Delta_y w^\circ(y)|^2 ds_y - {}^1/_2\mu^{-1}\sum_{\pm} h^{2\Lambda_\pm(\alpha)-3} k_\pm(\alpha, \nu)\, c_\pm^2 + O(h^2) \tag{6.3}$$

Here $\mathbf{E}_h(w^0)$ is the elastic energy of deformation of the Kirchhoff plate.

According to what was said before, when $\alpha \in (0, {}^1/_2\pi)$, the sum over $\pm$ vanishes from the asymptotic expression. If on the other hand $\alpha = \pi$, then by virtue of (3.4), (3.9) the integral along the contour $\partial\Omega$ will be divergent and relation (6.3) will need correcting. Let us carry out the corresponding calculations (they repeat, basically, the derivation /8/ of Eqs.(3.6)). Let us put $\Omega(\delta) = \{y \in \Omega : r > \delta\}$ and $\Gamma(\delta) = \{y \in \partial\Omega : r > \delta\}$, where $\delta > 0$. Remembering the special normalization of the corner part of (3.10) mentioned at the end of Sect.3, we find, using Green's formula, that

$$D_1^{-1} p w^1 dx = \lim_{\delta\to 0}\int_{\Omega(\delta)} w^1 \Delta_y^2 w^2\, dy =$$

$$\lim_{\delta\to 0}\int_{\partial\Omega(\delta)}\left(w^1\Delta_y\frac{\partial w^0}{\partial n} - \frac{\partial w^1}{\partial n}\Delta_y w^0 + \frac{\partial w^0}{\partial n}\Delta_y w^1 - w^0\Delta_y\frac{\partial w^1}{\partial n}\right) ds_y =$$

$$\lim_{\delta\to 0}\left\{\langle w^0, w^1\rangle_\delta - c(\nu)\int_{\Gamma(\delta)} |\Delta_y w^0(y)|^2\, ds_y\right\} =$$
$$-c(\nu)\lim_{\delta\to 0}\left\{8c_-^2 A^2 \ln\delta + \int_{\Gamma(\delta)} |\Delta_y w^0(y)|^2\, ds_y\right\} \equiv -c(\nu)\, L(w)$$

Since by virtue of (3.4) and (3.9) $|\Delta_y w^0(y)|^2 = 4r^{-1}(Ac_- \sin {}^1/_2\theta)^2 + O(r^{-1/2})$, it follows that the limit $L(w^0)$ exists. This means that, taking into account the term in (5.7) which is additional compared with (5.3), we obtain

$$U_h(u) = -{}^1/_2 h^{-3}\mathbf{E}_1(w^0) - {}^1/_2\mu^{-1}h^{-2}\ln hc(\nu)(1-\nu)^{-1}c_-^2 + {}^1/_2 h^{-2}\left[c(\nu)D_1 L(w^0) - \mu^{-1}\sum_{\pm} k_\pm(\pi,\nu)c_\pm^2\right] + O(h^{-1-\varepsilon}) \tag{6.4}$$

We stress that in order to find all terms of the asymptotic formula (6.3) or (6.4) which sharpens the classical formulas, we only need information concerning the solution w^0 of problem (2.1), and the values of three quantities $c(\nu)$ and $k_\pm(\alpha,\nu)$ characterizing the boundary layers. Just as in Sect.8 of /6/, we can obtain similar expressions for the eigenfrequencies of flexural oscillations of the plate.

We note, finally, that the method of matched asymptotic expansions (/12/ et al) enables us to construct a complete asymptotic series for solving problem (1.1)-(1.3) in the case of a load p smooth on $\bar{\Omega}$. Unlike series (2.7), this series contains terms of the three-dimensional boundary layer-type. Moreover, the power indices h are not integers and represent linear combinations (with integer coefficients) of the roots of Eq.(3.1).

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